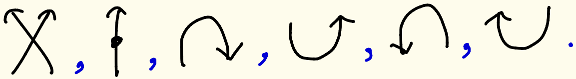
The Heisenberg category $\mathcal{H}eis_k$ $(k \in \mathbb{Z})$

$\mathbb{C}$-linear strict monoidal category.

Generators:

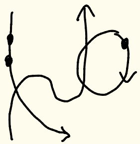
obs $\uparrow, \downarrow$

mors 

General ob = word

eg $\uparrow\uparrow\downarrow = \uparrow \otimes \uparrow \otimes \downarrow$

Penrose diagrams

eg  $: \uparrow\downarrow \to \downarrow\uparrow$

General mor = linear combination of diagrams $/\sim$

$\sim$ = rectilinear isotopy

Definition of $\mathcal{H}eis_k$ is due to

- Khovanov $(k=-1)$
- B.-Comes-Nash-Reynolds $(k=0)$ "affine oriented Brauer"
- Mackaay-Savage $(k<0)$
- B. $(k \in \mathbb{Z})$

$\exists$ Many other diagrammatic monoidal categories of a similar nature.

The Heisenberg category $\mathcal{H}eis_k$ $(k \in \mathbb{Z})$

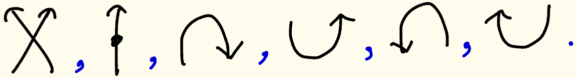
$\mathbb{C}$-linear strict monoidal category.

Generators: obs $\uparrow, \downarrow$

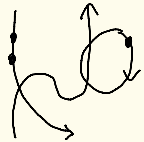
General ob = word

eg $\uparrow\uparrow\downarrow = \uparrow \otimes \uparrow \otimes \downarrow$

mors



eg

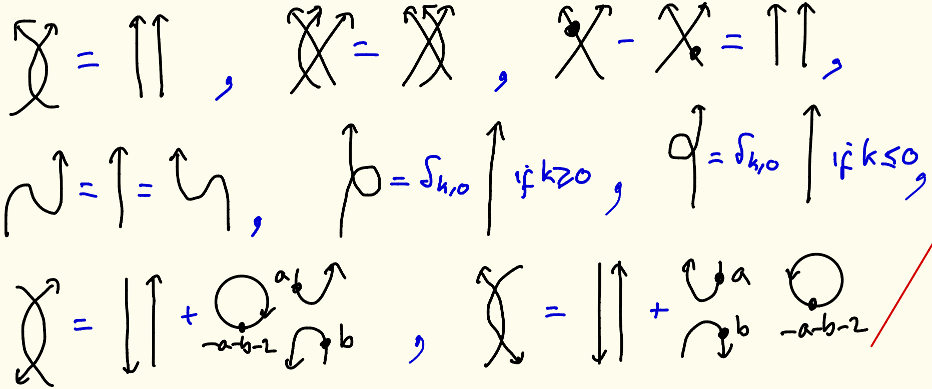


$: \uparrow\downarrow \to \downarrow\uparrow$

General mor = linear combination of diagrams / $\sim$

$\sim$ = rectilinear isotopy

Relations:



The Heisenberg category $\mathcal{H}eis_k$ $(k\in\mathbb{Z})$

$\mathbb{C}$-linear strict monoidal category.

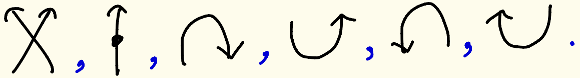
General ob = word

eg $\uparrow\uparrow\downarrow = \uparrow\otimes\uparrow\otimes\downarrow$

Generators:

obs: $\uparrow,\ \downarrow$

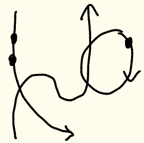
mors:



General mor = linear combination of diagrams $/\sim$

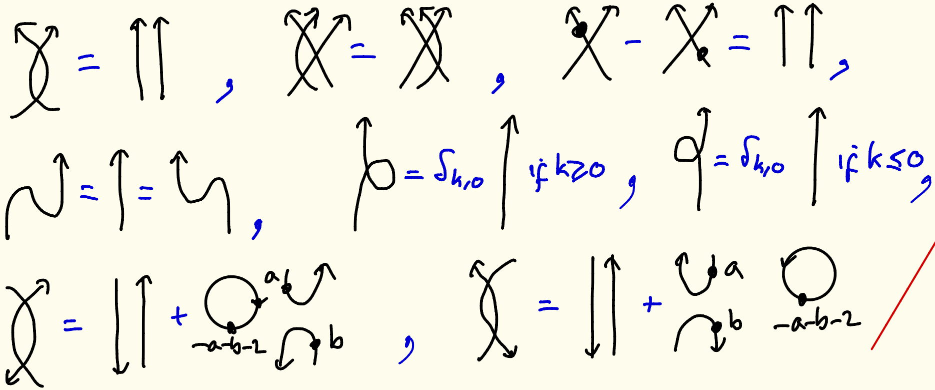
$\sim$ = rectilinear isotopy

eg



$: \uparrow\downarrow \to \downarrow\uparrow$

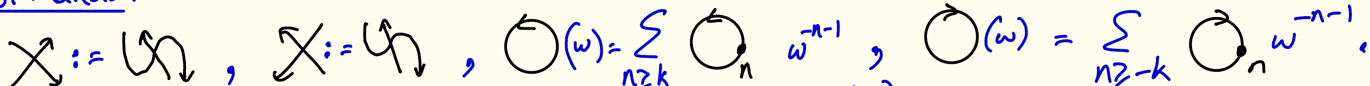
Relations:



degenerate affine Hecke algebra!

Negatively dotted bubbles are defined so

Shorthands:



(w = indeterminate)

Motivation: $\mathcal{H}eis_k \circlearrowright \bigoplus_{n \geq 0} H_n^f\text{-mod}$

degenerate cyclotomic Hecke algebra of level $\ell > 0$

- If $k = -\ell$

$(\uparrow, \downarrow) \longleftrightarrow (\mathrm{ind}_n^{n+1}, \mathrm{res}_{n-1}^n)$ adjoint pair!

- If $k = \ell$

$(\uparrow, \downarrow) \longleftrightarrow (\mathrm{res}_{n-1}^n, \mathrm{coind}_n^{n+1})$

$f = f(w) = (w - c_1) \cdots (w - c_\ell) \in \mathbb{C}[w]$

monic poly. of degree ℓ.

This was Khovanov's original idea — in level one $H_n^f = \mathbb{C}S_n$

$\mathcal{H}eis_k$ acts on many other important Abelian categories in rep. theory!

Motivation: $\mathcal{H}eis_k \circlearrowright \bigoplus_{n\geq 0} H_n^f\text{-mod}$

$\uparrow$ degenerate cyclotomic Hecke algebra of level $\ell > 0$

$f = f(w) = (w-c_1)\cdots(w-c_\ell) \in \mathbb{C}[w]$
monic poly. of degree ℓ.

- If $k = -\ell$ adjoint pair!
  $(\uparrow, \downarrow) \longleftrightarrow (\mathrm{ind}_n^{n+1}, \mathrm{res}_{n-1}^n)$
- If $k = \ell$
  $(\uparrow, \downarrow) \longleftrightarrow (\mathrm{res}_{n-1}^n, \mathrm{coind}_n^{n+1})$

Theorem 1 $K_0(\mathrm{Kar}(\mathcal{H}eis_k)) = \mathbb{Z}\text{-form for } \mathcal{U}(\mathfrak{h}) / \langle c = k \rangle,$
central reduction of enveloping algebra of ∞-dim Heisenberg Lie alg.

Originally conjectured by Khovanov.

Now a theorem — B., Savage, Webster

Motivation: $\mathcal{H}eis_k \circlearrowright \bigoplus_{n\geq 0} H_n^f\text{-mod}$

$\uparrow$ degenerate cyclotomic Hecke algebra of level $\ell > 0$

$f = f(w) = (w-c_1)\cdots(w-c_\ell) \in \mathbb{C}[w]$ monic poly. of degree ℓ.

adjoint pair!

- If $k = -\ell$: $(\uparrow, \downarrow) \longleftrightarrow (\mathrm{ind}_n^{n+1}, \mathrm{res}_{n-1}^n)$
- If $k = \ell$: $(\uparrow, \downarrow) \longleftrightarrow (\mathrm{res}_{n-1}^n, \mathrm{coind}_n^{n+1})$

Theorem 1 $K_0(\mathrm{Kar}(\mathcal{H}eis_k)) = \mathbb{Z}$-form for $U(\mathfrak{h}) / \langle c = k \rangle$, central reduction of enveloping algebra of ∞-dim Heisenberg Lie alg.

$\Uparrow$

Theorem 2 Basis theorem — $\mathrm{Hom}_{\mathcal{H}eis_k}(X,Y)$ is free right $\mathrm{End}_{\mathcal{H}eis_k}(\mathbb{1})$-module with basis = "reduced" diagrams w/ dots at termini ...

Bubbles (copy of Sym)

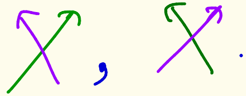
$\Uparrow$

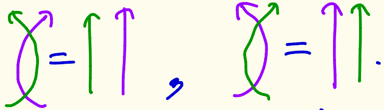
Theorem 3 Generic categorified comultiplication $\Delta_{\ell+m} : \mathcal{H}eis_{\ell+m} \to \mathcal{H}eis_\ell \,\bar{\odot}\, \mathcal{H}eis_m$

The categorical comultiplication

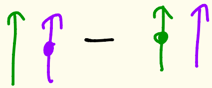
$\mathcal{H}eis_\ell \odot \mathcal{H}eis_m$ Symmetric product = monoidal category gen'd by both $\mathcal{H}eis_\ell$, $\mathcal{H}eis_m$ + morphisms forcing them to strictly commute

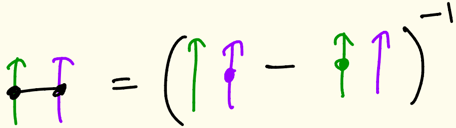
The categorical comultiplication

$\mathcal{H}eis_\ell \odot \mathcal{H}eis_n$ Use two colors! New morphisms 

New relations 

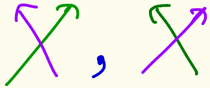
+ obvious commuting relations

$\mathcal{H}eis_\ell \bar{\odot} \mathcal{H}eis_n$ Localize at 

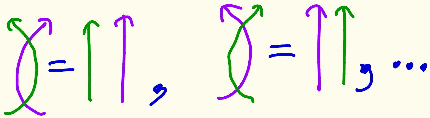
i.e. adjoin two-sided inverse 

The categorical comultiplication

$\mathcal{H}eis_\ell \odot \mathcal{H}eis_m$ Use two colors! New morphisms

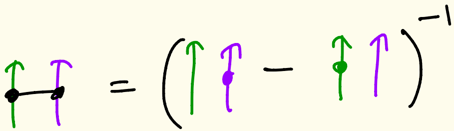


New relations

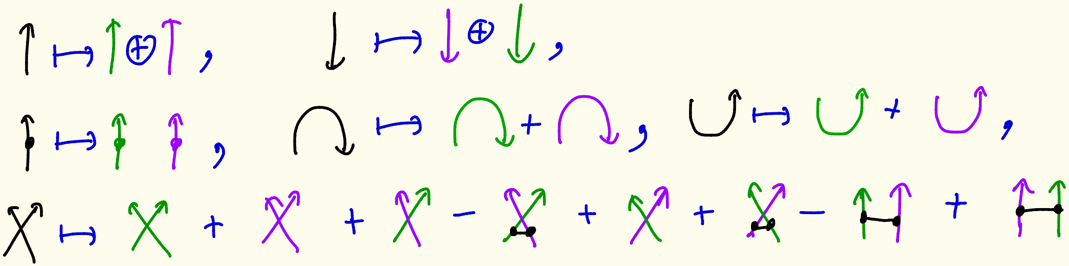


$\mathcal{H}eis_\ell \bar{\odot} \mathcal{H}eis_m$ Localize at

i.e. adjoin two-sided inverse



Theorem $\exists!$ strict monoidal functor $\Delta_{\ell+m} : \mathcal{H}eis_{\ell+m} \longrightarrow \mathrm{Add}(\mathcal{H}eis_\ell \bar{\odot} \mathcal{H}eis_m)$



Proof Work out images of cap, cup, then check relations!

Now take

$f(w) = w^\ell$ lineárized Cartesian product $g(w) = (w-c_1)\cdots(w-c_m)$

$$\left(\bigoplus_{m\geq 0} H_m^f\text{-mod}\right) \boxtimes \left(\bigoplus_{n\geq 0} H_n^g\text{-mod}\right)$$

$\mathcal{H}eis_{-\ell}$ acts $\mathcal{H}eis_m$ acts

Now take

$f(w) = w^\ell$ linearized Cartesian product $g(w) = (w-c_1)\cdots(w-c_m)$

$$\left(\bigoplus_{m\geq 0} H_m^f\text{-mod}\right) \boxtimes \left(\bigoplus_{n\geq 0} H_n^g\text{-mod}\right)$$

$\circlearrowright$ $\mathcal{H}eis_{-\ell}$ acts $\circlearrowright$ $\mathcal{H}eis_m$ acts

Providing $c_1,\dots,c_m \notin \mathbb{Z}$, $\uparrow \uparrow - \uparrow \uparrow$ acts invertibly ... so $\mathcal{H}eis_k$ acts via $\Delta_{-\ell+m}$

$k = m - \ell$

Moreover $\circlearrowleft(w)$ acts as $w^k + w^{k-1}e_1 + \cdots + w^{k-m}e_m$

$e_i := e_i(c_1,\dots,c_m)$

elementary symmetric functions

$\Delta_{-\ell+m}$

$\circlearrowleft(w) \quad \circlearrowleft(w)$

Now take

$f(w) = w^\ell$ $\qquad$ linearized Cartesian product $\qquad$ $g(w) = (w-c_1)\cdots(w-c_m)$

$$\left(\bigoplus_{m\geq 0} H^f_m\text{-mod}\right) \boxtimes \left(\bigoplus_{n\geq 0} H^g_n\text{-mod}\right)$$

Sufficiently large module category!

$\circlearrowleft$ $\mathrm{Heis}_{-\ell}$ acts $\qquad\qquad$ $\circlearrowleft$ Heis_m acts

Providing $c_1,\dots,c_m \notin \mathbb{Z}$, $\uparrow\uparrow - \uparrow\uparrow$ acts invertibly ... so Heis_k acts via $\Delta_{-\ell+m}$

$k = m - \ell$

Moreover $\bigcirc(w)$ acts as $w^k + w^{k-1}e_1 + \dots + w^{k-m}e_m$

$e_i := e_i(c_1,\dots,c_m)$

Sym

Letting $c_1,\dots,c_m$ vary, gives enough freedom to prove linear independence of bubbles,

$m, \ell >> 0$
$k = m - \ell$

hence the basis theorem! $\qquad$ <u>Model proof</u>

Thank you!