Combinatorial and Categorical Aspects of Representation Theory
San Francisco State University
San Francisco, CA

Representations of degenerate affine Brauer superalgebras

Mee Seong Im
United States Military Academy

October 28, 2018

Joint work.

This is a joint work with
Zajj Daugherty, Iva Halacheva, and Emily Norton.
(Preliminary report.)

Motivation.

Periplectic Lie superalgebras $\mathfrak{p}(n)$ form the first family of so-called "strange" Lie superalgebras in the classification of reductive Lie superalgebras.

In the process to construct higher Schur-Weyl duality for $\mathfrak{p}(n)$, non-semisimple algebras known as degenerate affine Brauer superalgebras (with defining parameter 0) sVW_a were constructed in order to be used as a tool in understanding the representation theory of $\mathfrak{p}(n)$.

The algebra sVW_a commutes with the action of $\mathfrak{p}(n)$ on the module $M \otimes V^{\otimes a}$.

Motivation.

- Our previous papers on periplectic Lie superalgebras established basic properties of a certain monoidal supercategory sVW and diagrammatic algebras sVW_a arising as spaces of morphisms in sVW.
- This project the four of us started during our two-week tenure at MSRI in July 2018, a natural continuation of that work, initiates the representation theory of the algebras sVW_a. These algebras have a diagrammatic presentation similar to affine versions of Brauer algebras.
- We study their finite-dimensional representations, which are representations of the cyclotomic quotients of sVW_a. In our work in progress, we have explicitly constructed the representations as matrices in small rank cases ($n \leq 25$).

Motivation.

- To tackle the general case, we plan to imitate the constructions of cellular algebras. We have an explicit description of the cell modules for which the first axiom of cellularity holds, and a conjectural labeling of the simple modules for any a.
- To finish the complete classification, it should suffice to adapt techniques used in similar situations by Coulembier, Bowman-Cox-De Visscher, and Elias, where cellularity does not literally hold but there is a technical work-around.

Degenerate affine Brauer superalgebras.

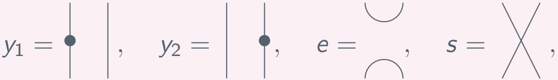
Work over $\mathbb{C}$, and assume $a = 2$.

The algebra sVW_2 is described as follows:

Generators: y_1, y_2, e, s

Relations:

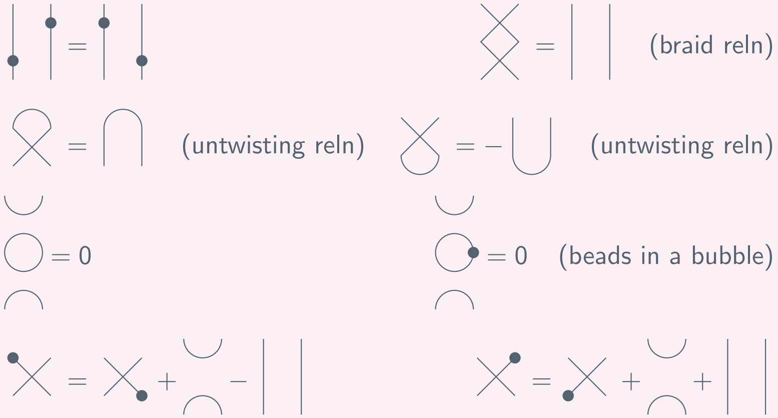
1. $y_1 y_2 = y_2 y_1$,
2. $s^2 = 1$,
3. $es = e$,
4. $se = -e$,
5. $e^2 = 0$,
6. $y_1 s = s y_2 + e - 1$,
7. $y_2 s = s y_1 + e + 1$,
8. $e y_1^{\ell} e = 0$ for all $\ell \geq 0$,
9. $e(y_2 - y_1) = e$,
10. $(y_2 - y_1)e = -e$.



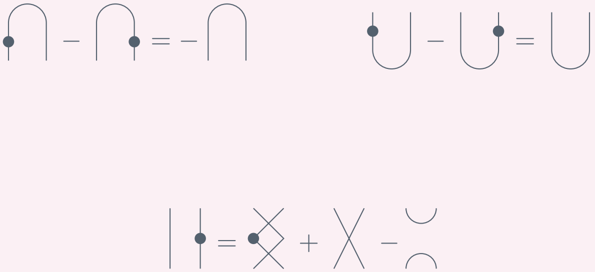
$y_1 =$,
$y_2 =$,
$e =$,
$s =$,

Degenerate affine Brauer superalgebras.

Relations.



Degenerate affine Brauer superalgebras.
Relations.



Calibrated and noncalibrated representations.

I will give an explicit description of a large family of irreducible modules of sVW_2.

A finite-dimensional module M is *calibrated* if y_is act semisimply.

In other words, a calibrated module is a module which has a basis of simultaneous eigenvectors for all the elements of a large commutative subalgebra inside sVW_2.

We are interested in constructing calibrated representations such that the action by the nilpotent generator e is nontrivial.

A finite-dimensional module M is *noncalibrated* if y_is do not act semisimply.

We are interested in constructing noncalibrated representations such that the action by e is nontrivial.

Calibrated representations.

The simple calibrated $s\mathbb{V}_2$-modules on which the nilpotent e acts trivially are precisely the Hecke algebra modules.

If V is an n-dimensional indecomposable calibrated $s\mathbb{V}_2$-module on which e acts nontrivially, then the generators of $s\mathbb{V}_2$ must be of the form

$$y_1 = \mathrm{diag}(y_1^{(1)}, y_2^{(1)}, \ldots, y_n^{(1)}), \quad y_2 = y_1 + \mathrm{diag}(\underbrace{-1, \ldots, -1}_{\nu}, \underbrace{1, \ldots, 1}_{n-\nu}),$$
$$e = \begin{pmatrix} 0 & 1_{\nu\times(n-\nu)} \\ 0 & 0 \end{pmatrix}, \quad s = \begin{pmatrix} -\mathrm{Id}_\nu & f(y_1)_{\nu\times(n-\nu)} \\ 0 & \mathrm{Id}_{n-\nu} \end{pmatrix}, \tag{1}$$

where $\mathrm{diag}(a_1, \ldots, a_n)$ is a diagonal matrix, e is an $n \times n$ matrix with 1 in the entries of $\nu \times (n-\nu)$ upper right block and zero elsewhere, Id_ν is a $\nu \times \nu$ identity matrix, and $f(y_1)_{\nu\times(n-\nu)}$ denotes that the entries of the upper right block of s are a function in coordinates of y_1.

There is also a condition on the eigenvalues $y_i^{(1)}$.

Noncalibrated representations.

Let V be an n-dimensional simple noncalibrated $s\mathbb{W}_2$-module. Then the generators of $s\mathbb{W}_2$ with respect to the basis of V must be of the form

$$y_1 = \begin{pmatrix} \lambda & 1 & 0 & \dots & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & \lambda & 1 \\ 0 & \dots & 0 & 0 & 0 & \lambda \end{pmatrix}, y_2 = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \dots & y_{1n} \\ 0 & y_{11} & y_{12} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & y_{13} \\ \vdots & \ddots & \ddots & \ddots & y_{12} \\ 0 & \dots & 0 & 0 & y_{11} \end{pmatrix},$$

$$e = 0, \qquad \text{and} \qquad s = (f(\lambda,\ y_{1,i},\ s_{1,j})). \tag{2}$$

There is a beautiful pattern (with a deep connection to Stanley's theorem) to describe s as a matrix representation.

Noncalibrated representations.

Combinatorial connections.

A connection between the number of the monomials with a positive coefficient in a coordinate function of s to Stanley's Theorem.

The number of distinct monomials with a positive coefficient in any k-th entry along level l superdiagonal of s is the total number of 1s that occur among all unordered partitions of the positive integer l, which is the sum of the numbers of distinct members of those partitions.

Example: The partitions of 3 are:

$$\{3\}, \quad \{1,2\}, \quad \{1,1,1\}.$$

There are a total of $0+1+3=4$ 1s in this list, which is equal to the sums of the numbers of unique terms in each partition $1+2+1=4$. Note that the number of 1s occurring in partitions of $1,2,3,4,\dots$ are

$$1, 2, 4, 7, 12, 19, 30, \dots.$$

Connections to the geometry of flag varieties and other directions.

Kazhdan-Lusztig construction of the irreducible representations of the affine Hecke algebras shows that the structure of these representations has deep connections to the geometry of certain subvarieties of the flag variety; it is a q-analogue of Springer's construction of the irreducible representations of the Weyl group on the cohomology of unipotent varieties.

It is our hope that these irreducible representations of sVW_2 also have connections to the geometry of certain subvarieties of the flag variety.

We also hope to index and give a combinatorial classification of calibrated and noncalibrated representations using Young tableaux of modified skew shapes.

Thank you.
Questions?