

NOTES ON THE \mathfrak{gl} -Web CATEGORY

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0.1. **Definition of \mathfrak{gl} -Web.** Let \mathbb{k} be a commutative ring with identity.

For any $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$, we use the generalized binomial coefficient

$$\binom{n}{k} := \frac{n(n-1)\cdots(n-k+1)}{k!} \in \mathbb{Z},$$

viewed as an element of \mathbb{k} .

Definition 0.1.1. Let $\mathfrak{gl}\text{-Web}$ denote a strict monoidal \mathbb{k} -linear category with generating objects $a \in \mathbb{Z}_{\geq 1}$. The monoidal structure is given on objects by concatenation. Thus an object in $\mathfrak{gl}\text{-Web}$ is a finite sequence of positive integers with the empty sequence being the unit object. More generally, we sometime choose to write objects of $\mathfrak{gl}\text{-Web}$ as a finite sequences of nonnegative integers where 0 indicates an instance of the unit object. The generating morphisms in this category are:

$$\begin{array}{ccc} \begin{array}{c} a \\ \backslash \quad / \\ a+b \end{array} & : a+b \rightarrow (a,b), & \begin{array}{c} a+b \\ \backslash \quad / \\ a \quad b \end{array} & : (a,b) \rightarrow a+b, \end{array}$$

for $a, b \in \mathbb{Z}_{\geq 0}$, where we write (a,b) for the monoidal product of the objects a and b , and a strand labelled by 0 is a morphism to or from the unit object. We call these morphisms *split*, and *merge*, respectively.

The following relations hold in $\mathfrak{gl}\text{-Web}$ for all $a, b, c \in \mathbb{Z}_{\geq 0}$:

Web-associativity:

$$\begin{array}{ccc} \begin{array}{c} a \quad b \quad c \\ \backslash \quad \backslash \quad \backslash \\ a+b+c \end{array} & = & \begin{array}{c} a \quad b \quad c \\ \backslash \quad \backslash \quad \backslash \\ a+b+c \end{array} & , & \begin{array}{c} a+b+c \\ \backslash \quad \backslash \quad \backslash \\ a \quad b \quad c \end{array} & = & \begin{array}{c} a+b+c \\ \backslash \quad \backslash \quad \backslash \\ b+c \end{array} ; \end{array} \tag{0.1}$$

Rung swap:

$$\begin{array}{ccc} \begin{array}{c} a-s+r \quad b+s-r \\ \backslash \quad / \\ a-s \quad b+s \\ \backslash \quad / \\ a \quad b \end{array} & = & \sum_{t \in \mathbb{Z}_{\geq 0}} \binom{a-b+r-s}{t} \begin{array}{c} a-s+r \quad b+s-r \\ \backslash \quad / \\ a+r-t \quad b-r+t \\ \backslash \quad / \\ a \quad b \end{array} & . \end{array} \tag{0.2}$$

If \mathbf{a} and \mathbf{b} are two sequences of nonnegative integers, then a general morphism in $\mathfrak{gl}\text{-Web}$ from \mathbf{a} to \mathbf{b} is a \mathbb{k} -linear combination of diagrams obtained by vertically and horizontally concatenating splits, merges, and vertical strands labeled by nonnegative integers and, moreover, when the labels along

the bottom (resp. top) of each diagram are read left to right one obtains the sequence \mathbf{a} (resp. \mathbf{b}). On diagrams composition is given by vertical concatenation and the monoidal structure is given by horizontal concatenation.

Going forward, and in the relations defined above, we use the following conventions:

- Strands labeled by ‘0’ are to be deleted;
- Diagrams containing a negatively-labeled strand are to be read as zero;
- We will sometimes choose to omit labels on strands when the label is clear from context.

For brevity, we also adopt the convention in calculations that when an equality follows from a previous result, this fact is indicated by placing the relevant equation number over the equals sign in question.

Remark 0.1.2. When \mathbb{k} is a field $\mathfrak{gl}\text{-Web}$ can be seen to be isomorphic to the *Schur category* defined in [1], which appeared as this paper was being prepared. As explained therein, the Schur category is related to the category introduced in [2]. Many of the relations established in the next section can be inferred from [1, 2] by using [1, Remark 4.8, Theorem 4.10], but we choose to provide proofs here because our starting assumptions are slightly different and in order to keep these notes self-contained.

0.2. Implied Relations for $\mathfrak{gl}\text{-Web}$. We first record a few relations which are implied by the defining relations of $\mathfrak{gl}\text{-Web}$. The following ‘knothole’ relation follows immediately from (0.2).

Lemma 0.2.1. *For all $a, b \in \mathbb{Z}_{\geq 0}$ we have*

$$\text{Diagram: } \begin{array}{c} a+b \\ \text{---} \text{---} \\ \text{---} \text{---} \\ a+b \end{array} = \binom{a+b}{a} \Bigg|_{a+b} .$$

Lemma 0.2.2. *One has*

$$\text{Diagram: } \begin{array}{c} a-s+r & b+s-r \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ a & b \end{array} = \sum_{t \in \mathbb{Z}_{\geq 0}} \binom{-a+b-r+s}{t} \begin{array}{c} a-s+r & b+s-r \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ a-s+t & b+s-t \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ a & b \end{array} ,$$

for all admissible $a, b, r, s \in \mathbb{Z}_{\geq 0}$.

Proof. For space, we write

$$L(x, y) := \begin{array}{c} a-y+x & b+y-x \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ a+x & b-x \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ a & b \end{array} \quad \text{and} \quad R(x, y) := \begin{array}{c} a-y+x & b+y-x \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ a-y & b+y \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ a & b \end{array} , \quad (0.3)$$

for any $x, y \in \mathbb{Z}_{\geq 0}$. We prove the lemma statement by induction on $n = r + s$. The claim is trivial for $n = 0, 1$, so let $n \geq 2$ and assume the claim holds for all $n' < n$. We have

$$\begin{aligned} L(r, s) &\stackrel{(0.2)}{=} R(r, s) - \sum_{w \in \mathbb{Z}_{>0}} \binom{a-b+r-s}{w} L(r-w, s-w) \\ &= R(r, s) - \sum_{w \in \mathbb{Z}_{>0}} \binom{a-b+r-s}{w} \sum_{u \in \mathbb{Z}_{\geq 0}} \binom{-a+b-r+s}{u} R(r-w-u, s-w-u) \end{aligned}$$

$$\begin{aligned}
&= R(r, s) - \sum_{t \in \mathbb{Z}_{>0}} \sum_{w=1}^t \binom{a-b+r-s}{w} \binom{-a+b-r+s}{t-w} R(r-t, s-t) \\
&= R(r, s) + \sum_{t \in \mathbb{Z}_{>0}} \binom{-a+b-r+s}{t} R(r-t, s-t),
\end{aligned}$$

where the second equality follows by the induction assumption, the last equality follows from the Chu-Vandermonde identity. This completes the proof. \square

Lemma 0.2.3. *The following equalities hold in $\mathfrak{gl}\text{-Web}$:*

$$\begin{aligned}
&\begin{array}{c} \text{Diagram with strands } s'' \text{ and } s' \text{ crossing at } b, \text{ and } s \text{ crossing at } a \text{ and } c. \end{array} = \sum_{t \in \mathbb{Z}_{\geq 0}} \binom{s-s'+s''}{t} \begin{array}{c} \text{Diagram with strands } s+s'' \text{ and } s''-t \text{ crossing at } b, \text{ and } s' \text{ crossing at } a \text{ and } c. \end{array}, \\
&\begin{array}{c} \text{Diagram with strands } r'' \text{ and } r' \text{ crossing at } b, \text{ and } r \text{ crossing at } a \text{ and } c. \end{array} = \sum_{t \in \mathbb{Z}_{\geq 0}} \binom{r-r'+r''}{t} \begin{array}{c} \text{Diagram with strands } r+r'' \text{ and } r''-t \text{ crossing at } b, \text{ and } r' \text{ crossing at } a \text{ and } c. \end{array},
\end{aligned}$$

for all admissible $a, b, c, r, r', r'', s, s', s'' \in \mathbb{Z}_{\geq 0}$.

Proof. The proof given in [3, Lemma 2.9], in the context of a web category where relations identical to (0.1) and (0.2) hold, is directly applicable to the $\mathfrak{gl}\text{-Web}$ category. \square

0.3. Braiding for $\mathfrak{gl}\text{-Web}$. We next establish the category $\mathfrak{gl}\text{-Web}$ admits a braiding.

For any $a, b \in \mathbb{Z}_{\geq 0}$, we define the crossing morphism:

$$\begin{array}{c} \text{Diagram with strands } a \text{ and } b \text{ crossing} \end{array} := \sum_{s-r=a-b} (-1)^{a-s} \begin{array}{c} \text{Diagram with strands } r \text{ and } s \text{ crossing at } b, \text{ and } a \text{ and } b+s \text{ crossing at } a \end{array} \stackrel{(0.2)}{=} \sum_{s-r=a-b} (-1)^{b-r} \begin{array}{c} \text{Diagram with strands } s \text{ and } r \text{ crossing at } a, \text{ and } a+r \text{ and } b-r \text{ crossing at } b \end{array}. \quad (0.4)$$

Lemma 0.3.1. *For all $a, b \in \mathbb{Z}_{\geq 0}$ we have*

$$\begin{array}{c} \text{Diagram with strands } a+b \text{ crossing} \end{array} = \begin{array}{c} \text{Diagram with strands } a+b \text{ crossing} \end{array} \quad \text{and} \quad \begin{array}{c} \text{Diagram with strands } a \text{ and } b \text{ crossing} \end{array} = \begin{array}{c} \text{Diagram with strands } a \text{ and } b \text{ crossing} \end{array}.$$

Proof. We prove the first equality. The second is similar. We have

$$\begin{array}{c} \text{Diagram with strands } a+b \text{ crossing} \end{array} = \sum_{s-r=a-b} (-1)^{a-s} \begin{array}{c} \text{Diagram with strands } r \text{ and } s \text{ crossing at } b, \text{ and } a \text{ and } b+s \text{ crossing at } a \end{array} \stackrel{(0.2)}{=} \sum_{s-r=a-b} (-1)^{a-s} \binom{b+s}{r} \begin{array}{c} \text{Diagram with strands } b+s \text{ and } s \text{ crossing at } b, \text{ and } a \text{ and } b \text{ crossing at } a \end{array}$$

$$\begin{aligned}
 & \stackrel{(0.1)}{=} \sum_{s-r=a-b} (-1)^{a-s} \binom{b+s}{r} \begin{array}{c} a+b \\ \text{---} \\ s \\ \text{---} \\ a \quad b \end{array} \\
 & \stackrel{(0.2)}{=} \sum_{s-r=a-b} (-1)^{a-s} \binom{b+s}{r} \binom{a}{s} \begin{array}{c} a+b \\ \text{---} \\ a \quad b \end{array}.
 \end{aligned}$$

Considering the coefficient in the last term and using the substitution $t := a - s$, we may write:

$$\sum_{s-r=a-b} (-1)^{a-s} \binom{b+s}{r} \binom{a}{s} = \sum_{t=0}^a (-1)^t \binom{b+a-t}{a} \binom{a}{t} = 1.$$

The last equality follows from an application of Euler's finite difference theorem (see [4, (10.13)]). This completes the proof. \square

The following theorem describes the basic relations involving the crossing morphism.

Theorem 0.3.2. *For all $a, b, c \in \mathbb{Z}_{\geq 0}$, we have:*

$$\begin{array}{ccc}
 \begin{array}{c} a \quad b \\ \text{---} \\ a \quad b \end{array} & = & \begin{array}{c} a \quad b \\ \text{---} \\ a \quad b \end{array} , \\
 \begin{array}{c} c \quad b \quad a \\ \text{---} \\ a \quad b \quad c \end{array} & = & \begin{array}{c} c \quad b \quad a \\ \text{---} \\ a \quad b \quad c \end{array} ,
 \end{array} \tag{0.5}$$

$$\begin{array}{ccc}
 \begin{array}{c} a+b \quad c \\ \text{---} \\ c \quad a \quad b \end{array} & = & \begin{array}{c} a+b \quad c \\ \text{---} \\ c \quad a \quad b \end{array} , \\
 \begin{array}{c} c \quad a+b \\ \text{---} \\ a \quad b \quad c \end{array} & = & \begin{array}{c} c \quad a+b \\ \text{---} \\ a \quad b \quad c \end{array} ,
 \end{array} \tag{0.6}$$

$$\begin{array}{ccc}
 \begin{array}{c} c \quad a \quad b \\ \text{---} \\ a+b \quad c \end{array} & = & \begin{array}{c} c \quad a \quad b \\ \text{---} \\ a+b \quad c \end{array} , \\
 \begin{array}{c} a \quad b \quad c \\ \text{---} \\ c \quad a+b \end{array} & = & \begin{array}{c} a \quad b \quad c \\ \text{---} \\ c \quad a+b \end{array} .
 \end{array} \tag{0.7}$$

We define a crossing morphism $\mathbf{a} \otimes \mathbf{b} \rightarrow \mathbf{b} \otimes \mathbf{a}$ for objects \mathbf{a} and \mathbf{b} by the following diagram:

$$\begin{array}{c}
 \begin{array}{ccccccc}
 b_1 & b_2 & \dots & b_s & a_1 & a_2 & \dots & a_r \\
 \text{---} & \text{---} & & \text{---} & \text{---} & \text{---} & & \text{---} \\
 \text{---} & \text{---} & & \text{---} & \text{---} & \text{---} & & \text{---} \\
 a_1 & a_2 & \dots & a_r & b_1 & b_2 & \dots & b_s
 \end{array} \\
 \begin{array}{c}
 \text{---} \\
 \text{---}
 \end{array}
 \end{array} \tag{0.8}$$

Once the previous theorem is proven it is straightforward to verify the following result.

Corollary 0.3.3. *The crossing morphisms defined in (0.8) define a symmetric braiding on \mathfrak{gl} -Web.*

0.4. Proof of the \mathfrak{gl} -Web Braiding Theorem. For ease of reading the proof of Theorem 0.3.2 is broken into the following lemmas.

Lemma 0.4.1. *For all $a, b \in \mathbb{Z}_{\geq 0}$, we have*

$$\begin{array}{ccc} \text{Diagram of a crossing} & = & \text{Two vertical lines} \\ a \quad b & & a \quad b \end{array}.$$

Proof. We have

$$\begin{aligned} \text{Diagram of a crossing} &= \sum_{\substack{s-r=a-b \\ s'-r'=b-a}} (-1)^{a+b-s-s'} \begin{array}{c} r' \\ s' \\ r \\ s \end{array} \\ &\stackrel{\text{L.0.2.2}}{=} \sum_{\substack{s-r=a-b \\ s'-r'=b-a \\ t \in \mathbb{Z}_{\geq 0}}} (-1)^{a+b-s-s'} \binom{s+s'}{t} \\ &\stackrel{\text{L.0.2.1}}{=} \sum_{\substack{s-r=a-b \\ s'-r'=b-a \\ t \in \mathbb{Z}_{\geq 0}}} (-1)^{a+b-s-s'} \binom{s+s'-t}{s'+a-b} \binom{s+s'-t}{s} \binom{s+s'}{t} R(s+s'-t, s+s'-t), \end{aligned}$$

where we use the fact that $s+s' = r+r'$ and $r' = s'-a+b$ in the last equality, and the notation of (0.3). Writing $m := s+s'-t$, $k := s+s'+a-b$, $c := b-a$, we may rewrite this as

$$\sum_{k,m \in \mathbb{Z}_{\geq 0}} (-1)^k \binom{c+k}{m} \sum_{s=0}^k \binom{m}{k-s} \binom{m}{s} R(m, m). \quad (0.9)$$

Now we may use the binomial identities

$$\binom{c+k}{m} = \sum_{u=0}^m \binom{c}{m-u} \binom{k}{u} \quad \text{and} \quad \sum_{s=0}^k \binom{m}{k-s} \binom{m}{s} = \binom{2m}{k}$$

to rewrite (0.9) as

$$\begin{aligned} &\sum_{m \in \mathbb{Z}_{\geq 0}} \sum_{u=0}^m \binom{c}{m-u} \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} \binom{k}{u} R(m, m) \\ &= \sum_{m \in \mathbb{Z}_{\geq 0}} \sum_{u=0}^m \binom{c}{m-u} \sum_{k=0}^{2m} (-1)^k \binom{2m}{u} \binom{2m-u}{k-u} R(m, m) \\ &= \sum_{m \in \mathbb{Z}_{\geq 0}} \sum_{u=0}^m \binom{2m}{u} \binom{c}{m-u} \sum_{\ell=0}^{2m-u} (-1)^\ell \binom{2m-u}{\ell} R(m, m) \\ &= R(0, 0), \end{aligned}$$

since the binomial theorem implies

$$\sum_{\ell=0}^{2m-u} (-1)^\ell \binom{2m-u}{\ell} = 0$$

whenever $m > 0$. □

Lemma 0.4.2. *Crossings intertwine splits and merges:*

$$\begin{array}{ccc}
 \text{Diagram 1: } & \text{Diagram 2: } & \text{Diagram 3: } \\
 \begin{array}{c} a+b \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ c \quad a \quad b \end{array} & = & \begin{array}{c} a+b \quad c \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ c \quad a \quad b \end{array}, \\
 \begin{array}{c} c \quad a \quad b \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ a+b \quad c \end{array} & = & \begin{array}{c} c \quad a+b \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ a \quad b \quad c \end{array}, \\
 \begin{array}{c} c \quad a \quad b \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ a+b \quad c \end{array} & = & \begin{array}{c} c \quad a+b \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ a \quad b \quad c \end{array}, \\
 \begin{array}{c} c \quad a \quad b \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ c \quad a+b \end{array} & = & \begin{array}{c} a \quad b \quad c \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ c \quad a+b \end{array},
 \end{array}$$

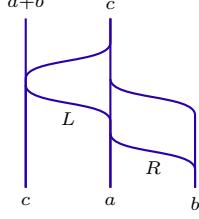
for all $a, b, c \in \mathbb{Z}_{\geq 0}$.

Proof. We prove the first equality. The others are proved via analogous arguments. We have

$$\begin{array}{c}
 \text{Diagram 1: } \begin{array}{c} a+b \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ c \quad a \quad b \end{array} = \sum_{\substack{s-r=c-a \\ s'-r'=c-b}} (-1)^{s+s'} \text{Diagram 2: } \begin{array}{c} a+b \quad c \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ c \quad a \quad b \\ \text{Diagram 3: } \begin{array}{c} a+b \quad c \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ c \quad a \quad b \end{array} \\
 \stackrel{(0.1)}{=} \sum_{\substack{s-r=c-a \\ s'-r'=c-b}} (-1)^{s+s'} \text{Diagram 4: } \begin{array}{c} a+b \quad c \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ c \quad a \quad b \\ \text{Diagram 5: } \begin{array}{c} a+b \quad c \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ c \quad a \quad b \end{array} \\
 \stackrel{\text{L.0.2.3}}{=} \sum_{\substack{s-r=c-a \\ s'-r'=c-b}} (-1)^{s+s'} \binom{a+s-s'}{r} \text{Diagram 6: } \begin{array}{c} a+b \quad c \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ c \quad a \quad b \\ \text{Diagram 7: } \begin{array}{c} a+b \quad c \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ c \quad a \quad b \end{array} \\
 \stackrel{(0.2)}{=} \sum_{\substack{s-r=c-a \\ s'-r'=c-b \\ t \in \mathbb{Z}_{\geq 0}}} (-1)^{s+s'} \binom{a+s-s'}{r} \binom{r}{t} \text{Diagram 8: } \begin{array}{c} a+b \quad c \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ c \quad a \quad b \\ \text{Diagram 9: } \begin{array}{c} a+b \quad c \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ c \quad a \quad b \end{array} \\
 \stackrel{(0.1)}{=} \sum_{\substack{s-r=c-a \\ s'-r'=c-b \\ t \in \mathbb{Z}_{\geq 0}}} (-1)^{s+s'} \binom{a+s-s'}{r} \binom{r}{t} \text{Diagram 10: } \begin{array}{c} a+b \quad c \\ \diagup \quad \diagdown \\ \text{crossing} \\ \diagdown \quad \diagup \\ c \quad a \quad b \end{array}
 \end{array}
 \end{array}$$

$$\begin{aligned}
 \text{L.0.2.3} \quad & \sum_{\substack{s-r=c-a \\ s'-r'=c-b \\ t \in \mathbb{Z}_{\geq 0} \\ u \in \mathbb{Z}_{\geq 0}}} (-1)^{s+s'} \binom{a+s-s'}{r} \binom{r}{t} \binom{c-s'+t}{u} \\
 & = \sum_{\substack{s-r=c-a \\ s'-r'=c-b \\ t \in \mathbb{Z}_{\geq 0} \\ u \in \mathbb{Z}_{\geq 0}}} (-1)^{s+s'} \binom{a+s-s'}{r} \binom{r}{t} \binom{c-s'+t}{u} \\
 & \quad \begin{array}{c} \text{Diagram showing two vertical lines } a+b \text{ and } c. \text{ The left line } a+b \text{ has labels } c \text{ at bottom and } a+b \text{ at top. The right line } c \text{ has labels } c \text{ at top and } a+b \text{ at bottom. Between them are two blue wavy lines. The top wavy line has labels } c-s-u \text{ and } r-t \text{ on its right. The bottom wavy line has labels } r+r' \text{ and } -t+u \text{ on its left.} \\ \text{Diagram showing two vertical lines } a+b \text{ and } c. \text{ The left line } a+b \text{ has labels } c \text{ at bottom and } a+b \text{ at top. The right line } c \text{ has labels } c \text{ at top and } a+b \text{ at bottom. Between them are two blue wavy lines. The top wavy line has labels } r+r' \text{ and } -t+u \text{ on its left. The bottom wavy line has labels } r'-t \text{ and } a \text{ on its right.} \end{array}
 \end{aligned}$$

Now, fixing $L, R \in \mathbb{Z}_{\geq 0}$, we consider the coefficient $\kappa(L, R)$ of



in the sum above. Using the substitutions $A = L - R - a + c - s$, $B = c - s'$, we have

$$\begin{aligned}
 \kappa(L, R) & = (-1)^{L+R+a} \sum_{B=0}^{b-R} \sum_{A=0}^{b-R} (-1)^{A+B} \binom{L-R-A+B}{B} \binom{L-R-A}{b-R-B} \binom{b-R}{A} \\
 & = \sum_{B=0}^{b-R} \sum_{A=0}^{b-R} (-1)^{a+A+B} \binom{b-R}{B} \binom{L-R-A+B}{b-R} \binom{b-R}{A} \\
 & = (-1)^{L+R+a} \sum_{B=0}^{b-R} (-1)^B \binom{b-R}{B} \sum_{A=0}^{b-R} (-1)^A \binom{L-R-A+B}{b-R} \binom{b-R}{A}.
 \end{aligned}$$

An application of Euler's finite difference theorem (cf. [4, (10.13)]), shows that

$$\sum_{A=0}^{b-R} (-1)^A \binom{L-R-A+B}{b-R} \binom{b-R}{A} = 1,$$

so

$$\kappa(L, R) = (-1)^{L+R+a} \sum_{B=0}^{b-R} (-1)^B \binom{b-R}{B} = (-1)^{L+a+b} \delta_{b,R},$$

by the binomial theorem. Therefore, we have

$$\begin{array}{ccc}
 \text{Diagram: } & & \text{Diagram: } \\
 \text{Left: } & & \text{Middle: } \\
 \text{Right: } & & \text{Right: }
 \end{array}$$

completing the proof. \square

Lemma 0.4.3. *Crossings satisfy the braid relation:*

$$\begin{array}{ccc}
 \text{Diagram: } & & \text{Diagram: } \\
 \text{Left: } & = & \text{Right: }
 \end{array}$$

for all $a, b, c \in \mathbb{Z}_{\geq 0}$.

Proof. We have

$$\begin{array}{ccc}
 \text{Diagram: } & & \text{Diagram: } \\
 \text{Left: } & = & \text{Middle: } \\
 & \sum_{s-r=a-c} (-1)^{a-s} & \text{L.0.4.2: } \sum_{s-r=a-c} (-1)^{a-s} \\
 & & \text{Right: } \\
 & & \text{Diagram: } \\
 & & \text{L.0.4.1: } \sum_{s-r=a-c} (-1)^{a-s} \\
 & & \text{Diagram: } \\
 & & \text{L.0.4.1: } \sum_{s-r=a-c} (-1)^{a-s} \\
 & & \text{Diagram: } \\
 & & = \text{Diagram: }
 \end{array}$$

as desired. \square

0.5. **A Useful Identity.** Using the crossing we record an identity in $\mathfrak{gl}\text{-Web}$ which will be useful in later calculations.

Lemma 0.5.1. *For all $a, b, c, d \in \mathbb{Z}_{\geq 0}$ such that $a + b = c + d$, we have*

$$\begin{array}{c} c \\ \backslash \quad / \\ a \quad b \end{array} = \sum_{t \in \mathbb{Z}_{\geq 0}} \begin{array}{c} c \\ \backslash \quad / \\ a-t \quad t \\ \backslash \quad / \\ a \quad b \end{array} \quad \begin{array}{c} d \\ \backslash \quad / \\ b \\ \backslash \quad / \\ a \end{array} = \sum_{t \in \mathbb{Z}_{\geq 0}} \begin{array}{c} c \\ \backslash \quad / \\ a-t \quad t \\ \backslash \quad / \\ a \quad b \end{array} \quad \begin{array}{c} d \\ \backslash \quad / \\ b-t \\ \backslash \quad / \\ b-a-t \\ \backslash \quad / \\ a \quad b \end{array} .$$

Proof. We go by induction on $n := a + b$. The base case $n = 0$ is trivial. Fix $n = a + b$ and assume the claim holds for all $a' + b' < n$. We first consider the case $b = c$. We have

$$\begin{array}{c} b \\ \backslash \quad / \\ a \\ \backslash \quad / \\ a \end{array} \stackrel{(0.4)}{=} \begin{array}{c} b \\ \backslash \quad / \\ a \\ \backslash \quad / \\ a \end{array} + \sum_{t \in \mathbb{Z}_{>0}} (-1)^{t+1} \begin{array}{c} b \\ \backslash \quad / \\ b-t \\ \backslash \quad / \\ a-t \quad b+a-t \\ \backslash \quad / \\ a \quad b \end{array}$$

$$= \begin{array}{c} b \\ \backslash \quad / \\ a \\ \backslash \quad / \\ a \end{array} + \sum_{\substack{t \in \mathbb{Z}_{>0} \\ u \in \mathbb{Z}_{\geq 0}}} (-1)^{t+1} \begin{array}{c} b \\ \backslash \quad / \\ b-t \\ \backslash \quad / \\ a-t \quad u \\ \backslash \quad / \\ a \quad b \end{array}$$

$$\stackrel{(0.1,0.2)}{=} \begin{array}{c} b \\ \backslash \quad / \\ a \\ \backslash \quad / \\ a \end{array} + \sum_{\substack{t \in \mathbb{Z}_{>0} \\ u \in \mathbb{Z}_{\geq 0}}} (-1)^{t+1} \binom{a-u}{t} \begin{array}{c} b \\ \backslash \quad / \\ u \\ \backslash \quad / \\ a-u \quad b-u \\ \backslash \quad / \\ a \quad b \end{array} ,$$

where we have applied the induction assumption in the second line. Then for fixed $u \leq a$, we have

$$\sum_{t \in \mathbb{Z}_{>0}} (-1)^{t+1} \binom{a-u}{t} = \begin{cases} 0 & \text{if } u = a \\ 1 & \text{if } u < a, \end{cases}$$

giving the result.

Now assume that $b > c$. Then we have

$$\begin{array}{c} c \\ \backslash \quad / \\ a \quad b \end{array} \stackrel{(0.2)}{=} \sum_{t \in \mathbb{Z}_{\geq 0}} \binom{c-b}{t} \begin{array}{c} c \\ \backslash \quad / \\ a-t \quad c-t \\ \backslash \quad / \\ a \quad b \end{array}$$

$$\begin{aligned}
&= \sum_{t,u \in \mathbb{Z}_{\geq 0}} \binom{c-b}{t} \begin{array}{c} \text{Diagram of a vertical rectangle with vertices } (b, b), (c, b), (c, d), (b, d). \text{ A blue line segment connects } (b, u) \text{ to } (c, u). \text{ A blue line segment connects } (c, a-t) \text{ to } (c, d). \text{ A blue line segment connects } (b, c-t) \text{ to } (c, c-t). \end{array} \\
&\stackrel{(0.1,0.2)}{=} \sum_{t,u \in \mathbb{Z}_{\geq 0}} \binom{c-b}{t} \binom{d-u}{b-c+t} \begin{array}{c} \text{Diagram of a vertical rectangle with vertices } (a, b), (c, b), (c, d), (a, d). \text{ A blue line segment connects } (a, u) \text{ to } (c, u). \text{ A blue line segment connects } (a, d-u) \text{ to } (c, d-u). \end{array}
\end{aligned}$$

where we have used the induction assumption in the second line. Now, for a fixed $u \leq d$, we have

$$\sum_{t \in \mathbb{Z}_{\geq 0}} \binom{c-b}{t} \binom{d-u}{b-c+t} = 1,$$

by an application of the index shift formula (see [4, (6.69)]), which yields the result. \square

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