NOTES ON THE $\mathfrak{gl}$-Web CATEGORY

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0.1. Definition of $\mathfrak{gl}$-Web.

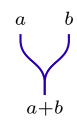
Let $\Bbbk$ be a commutative ring with identity.

For any $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$, we use the generalized binomial coefficient

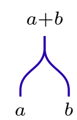
$$\binom{n}{k} := \frac{n(n-1)\cdots(n-k+1)}{k!} \in \mathbb{Z},$$

viewed as an element of $\Bbbk$.

Definition 0.1.1. Let $\mathfrak{gl}$**-Web** denote a strict monoidal $\Bbbk$-linear category with generating objects $a \in \mathbb{Z}_{\geq 1}$. The monoidal structure is given on objects by concatenation. Thus an object in $\mathfrak{gl}$**-Web** is a finite sequence of positive integers with the empty sequence being the unit object. More generally, we sometime choose to write objects of $\mathfrak{gl}$**-Web** as a finite sequences of nonnegative integers where 0 indicates an instance of the unit object. The generating morphisms in this category are:



$: a + b \to (a, b),$

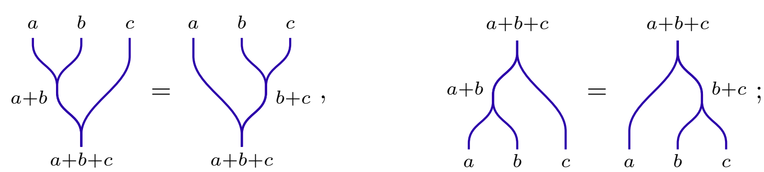


$: (a, b) \to a + b,$

for $a, b \in \mathbb{Z}_{\geq 0}$, where we write (a, b) for the monoidal product of the objects a and b, and a strand labelled by 0 is a morphism to or from the unit object. We call these morphisms *split*, and *merge*, respectively.

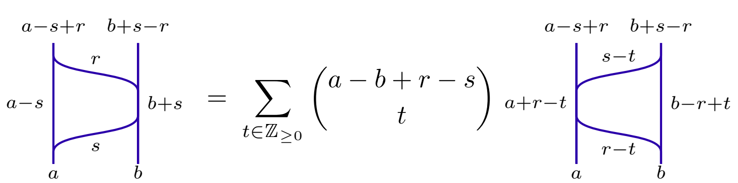
The following relations hold in $\mathfrak{gl}$**-Web** for all $a, b, c \in \mathbb{Z}_{\geq 0}$; :

Web-associativity:



(0.1)

Rung swap:



$$\sum_{t \in \mathbb{Z}_{\geq 0}} \binom{a-b+r-s}{t} \tag{0.2}$$

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two sequences of nonnegative integers, then a general morphism in $\mathfrak{gl}$**-Web** from $\boldsymbol{a}$ to $\boldsymbol{b}$ is a $\Bbbk$-linear combination of diagrams obtained by vertically and horizontally concatenating splits, merges, and vertical strands labeled by nonnegative integers and, moreover, when the labels along

Date: July 16, 2021.

the bottom (resp. top) of each diagram are read left to right one obtains the sequence $\boldsymbol{a}$ (resp. $\boldsymbol{b}$). On diagrams composition is given by vertical concatenation and the monoidal structure is given by horizontal concatenation.

Going forward, and in the relations defined above, we use the following conventions:

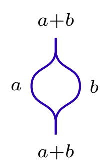
- Strands labeled by '0' are to be deleted;
- Diagrams containing a negatively-labeled strand are to be read as zero;
- We will sometimes choose to omit labels on strands when the label is clear from context.

For brevity, we also adopt the convention in calculations that when an equality follows from a previous result, this fact is indicated by placing the relevant equation number over the equals sign in question.

Remark 0.1.2. When $\Bbbk$ is a field $\mathfrak{gl}$**-Web** can be seen to be isomorphic to the *Schur category* defined in [1], which appeared as this paper was being prepared. As explained therein, the Schur category is related to the category introduced in [2]. Many of the relations established in the next section can be inferred from [1, 2] by using [1, Remark 4.8, Theorem 4.10], but we choose to provide proofs here because our starting assumptions are slightly different and in order to keep these notes self-contained.

0.2. **Implied Relations for $\mathfrak{gl}$-Web.** We first record a few relations which are implied by the defining relations of $\mathfrak{gl}$**-Web**. The following 'knothole' relation follows immediately from (0.2).

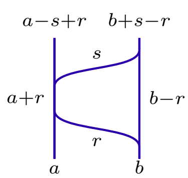
Lemma 0.2.1. *For all $a, b \in \mathbb{Z}_{\geq 0}$ we have*

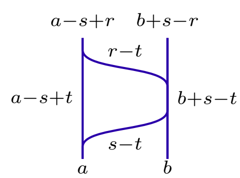


$$= \binom{a+b}{a}$$


.

Lemma 0.2.2. *One has*

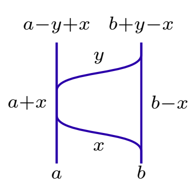


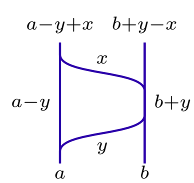
$$= \sum_{t \in \mathbb{Z}_{\geq 0}} \binom{-a+b-r+s}{t}$$


,

for all admissible $a, b, r, s \in \mathbb{Z}_{\geq 0}$.

Proof. For space, we write

$L(x,y) :=$


and $R(x,y) :=$


, (0.3)

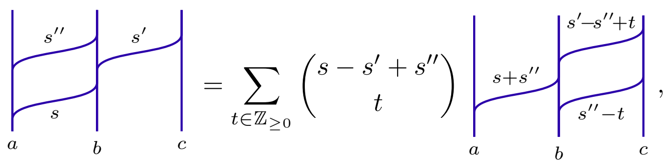
for any $x, y \in \mathbb{Z}_{\geq 0}$. We prove the lemma statement by induction on $n = r + s$. The claim is trivial for $n = 0, 1$, so let $n \geq 2$ and assume the claim holds for all $n' < n$. We have

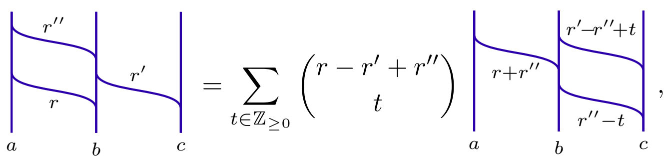
$$\begin{aligned} L(r,s) &\overset{(0.2)}{=} R(r,s) - \sum_{w \in \mathbb{Z}_{>0}} \binom{a-b+r-s}{w} L(r-w, s-w) \\ &= R(r,s) - \sum_{w \in \mathbb{Z}_{>0}} \binom{a-b+r-s}{w} \sum_{u \in \mathbb{Z}_{\geq 0}} \binom{-a+b-r+s}{u} R(r-w-u, s-w-u) \end{aligned}$$

$$= R(r,s) - \sum_{t\in\mathbb{Z}_{>0}} \sum_{w=1}^{t} \binom{a-b+r-s}{w}\binom{-a+b-r+s}{t-w} R(r-t,s-t)$$
$$= R(r,s) + \sum_{t\in\mathbb{Z}_{>0}} \binom{-a+b-r+s}{t} R(r-t,s-t),$$

where the second equality follows by the induction assumption, the last equality follows from the Chu-Vandermonde identity. This completes the proof. ☐

Lemma 0.2.3. *The following equalities hold in* $\mathfrak{gl}$-**Web***:*



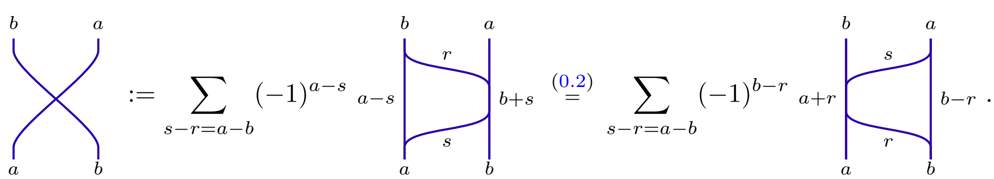


for all admissible $a,b,c,r,r',r'',s,s',s''\in\mathbb{Z}_{\geq 0}$.

Proof. The proof given in [3, Lemma 2.9], in the context of a web category where relations identical to (0.1) and (0.2) hold, is directly applicable to the $\mathfrak{gl}$-**Web** category. ☐

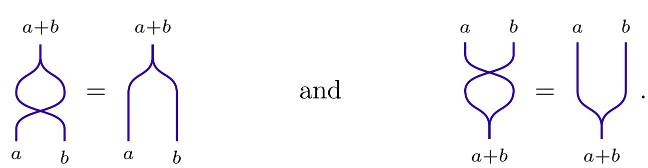
0.3. **Braiding for $\mathfrak{gl}$-Web.** We next establish the category $\mathfrak{gl}$-**Web** admits a braiding.

For any $a,b\in\mathbb{Z}_{\geq 0}$, we define the crossing morphism:

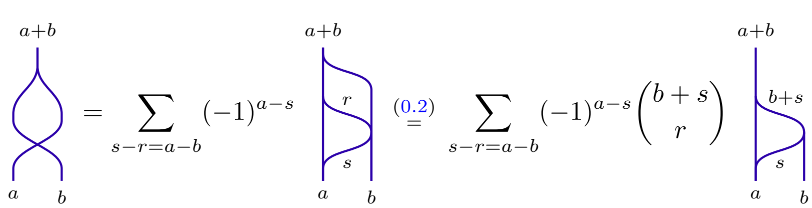


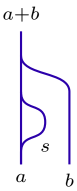
(0.4)

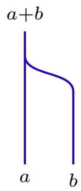
Lemma 0.3.1. *For all* $a,b\in\mathbb{Z}_{\geq 0}$ *we have*



Proof. We prove the first equality. The second is similar. We have



$$\overset{(0.1)}{=} \sum_{s-r=a-b} (-1)^{a-s} \binom{b+s}{r}$$ 

$$\overset{(0.2)}{=} \sum_{s-r=a-b} (-1)^{a-s} \binom{b+s}{r}\binom{a}{s}$$ 

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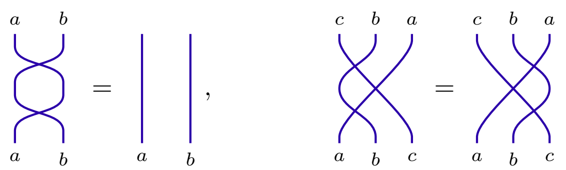
Considering the coefficient in the last term and using the substitution $t := a - s$, we may write:

$$\sum_{s-r=a-b} (-1)^{a-s} \binom{b+s}{r}\binom{a}{s} = \sum_{t=0}^{a} (-1)^t \binom{b+a-t}{a}\binom{a}{t} = 1.$$

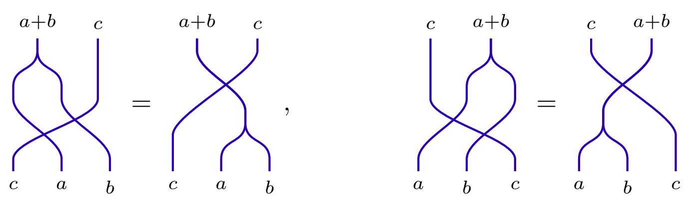
The last equality follows from an application of Euler's finite difference theorem (see [4, (10.13)]). This completes the proof. ☐

The following theorem describes the basic relations involving the crossing morphism.

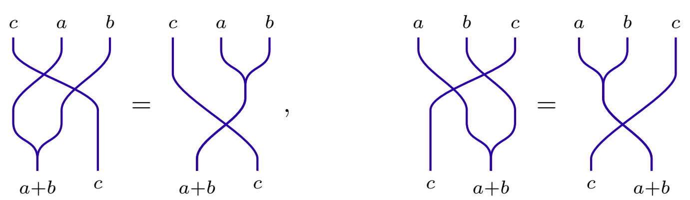
Theorem 0.3.2. *For all $a, b, c \in \mathbb{Z}_{\geq 0}$, we have:*



(0.5)

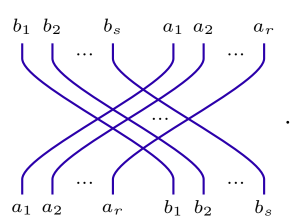


(0.6)



(0.7)

We define a crossing morphism $\boldsymbol{a} \otimes \boldsymbol{b} \to \boldsymbol{b} \otimes \boldsymbol{a}$ for objects $\boldsymbol{a}$ and $\boldsymbol{b}$ by the following diagram:



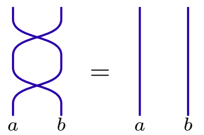
. (0.8)

Once the previous theorem is proven it is straightforward to verify the following result.

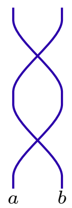
Corollary 0.3.3. *The crossing morphisms defined in (0.8) define a symmetric braiding on* $\mathfrak{gl}$-**Web**.

0.4. **Proof of the $\mathfrak{gl}$-Web Braiding Theorem.** For ease of reading the proof of Theorem 0.3.2 is broken into the following lemmas.

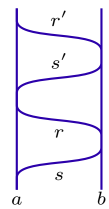
Lemma 0.4.1. *For all $a, b \in \mathbb{Z}_{\geq 0}$, we have*



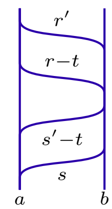
Proof. We have



$$= \sum_{\substack{s-r=a-b \\ s'-r'=b-a}} (-1)^{a+b-s-s'}$$



$$\overset{\text{L.0.2.2}}{=} \sum_{\substack{s-r=a-b \\ s'-r'=b-a \\ t\in\mathbb{Z}_{\geq 0}}} (-1)^{a+b-s-s'} \binom{s+s'}{t}$$



$$\overset{\text{L.0.2.1}}{=} \sum_{\substack{s-r=a-b \\ s'-r'=b-a \\ t\in\mathbb{Z}_{\geq 0}}} (-1)^{a+b-s-s'} \binom{s+s'-t}{s'+a-b}\binom{s+s'-t}{s}\binom{s+s'}{t} R(s+s'-t, s+s'-t),$$

where we use the fact that $s + s' = r + r'$ and $r' = s' - a + b$ in the last equality, and the notation of (0.3). Writing $m := s + s' - t$, $k := s + s' + a - b$, $c := b - a$, we may rewrite this as

$$\sum_{k,m\in\mathbb{Z}_{\geq 0}} (-1)^k \binom{c+k}{m} \sum_{s=0}^{k} \binom{m}{k-s}\binom{m}{s} R(m,m). \tag{0.9}$$

Now we may use the binomial identities

$$\binom{c+k}{m} = \sum_{u=0}^{m} \binom{c}{m-u}\binom{k}{u} \qquad \text{and} \qquad \sum_{s=0}^{k} \binom{m}{k-s}\binom{m}{s} = \binom{2m}{k}$$

to rewrite (0.9) as

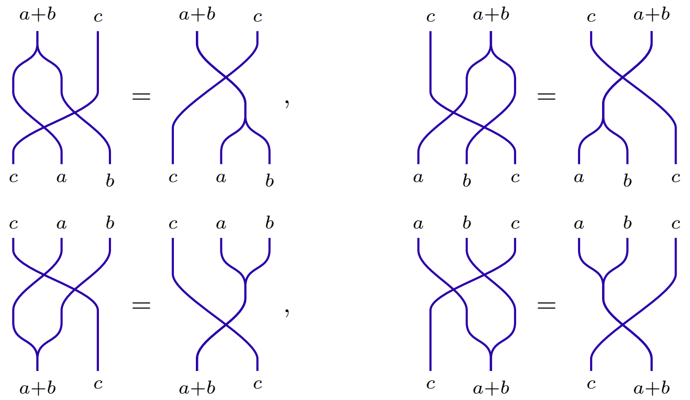
$$\begin{aligned}
&\sum_{m\in\mathbb{Z}_{\geq 0}} \sum_{u=0}^{m} \binom{c}{m-u} \sum_{k=0}^{2m} (-1)^k \binom{2m}{k}\binom{k}{u} R(m,m) \\
&\qquad = \sum_{m\in\mathbb{Z}_{\geq 0}} \sum_{u=0}^{m} \binom{c}{m-u} \sum_{k=0}^{2m} (-1)^k \binom{2m}{u}\binom{2m-u}{k-u} R(m,m) \\
&\qquad = \sum_{m\in\mathbb{Z}_{\geq 0}} \sum_{u=0}^{m} \binom{2m}{u}\binom{c}{m-u} \sum_{\ell=0}^{2m-u} (-1)^\ell \binom{2m-u}{\ell} R(m,m) \\
&\qquad = R(0,0),
\end{aligned}$$

since the binomial theorem implies

$$\sum_{\ell=0}^{2m-u} (-1)^\ell \binom{2m-u}{\ell} = 0$$

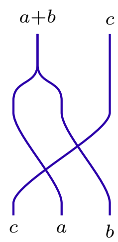
whenever $m > 0$. □

Lemma 0.4.2. *Crossings intertwine splits and merges:*

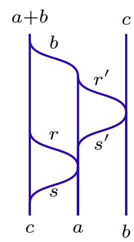


for all $a, b, c \in \mathbb{Z}_{\geq 0}$.

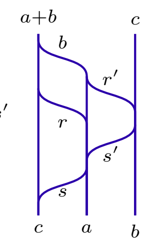
Proof. We prove the first equality. The others are proved via analogous arguments. We have



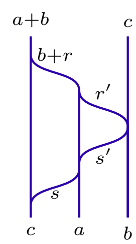
$$= \sum_{\substack{s-r=c-a \\ s'-r'=c-b}} (-1)^{s+s'}$$



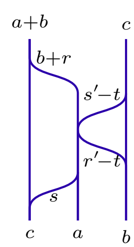
$$\overset{(0.1)}{=} \sum_{\substack{s-r=c-a \\ s'-r'=c-b}} (-1)^{s+s'}$$



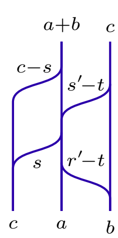
$$\overset{\text{L.0.2.3}}{=} \sum_{\substack{s-r=c-a \\ s'-r'=c-b}} (-1)^{s+s'} \binom{a+s-s'}{r}$$



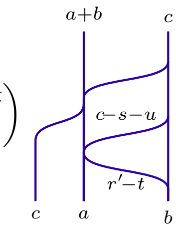
$$\overset{(0.2)}{=} \sum_{\substack{s-r=c-a \\ s'-r'=c-b \\ t\in\mathbb{Z}_{\geq 0}}} (-1)^{s+s'} \binom{a+s-s'}{r} \binom{r}{t}$$



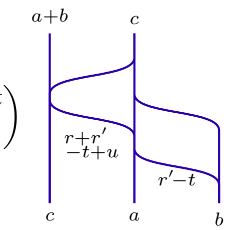
$$\overset{(0.1)}{=} \sum_{\substack{s-r=c-a \\ s'-r'=c-b \\ t\in\mathbb{Z}_{\geq 0}}} (-1)^{s+s'} \binom{a+s-s'}{r} \binom{r}{t}$$



$$\overset{\text{L.0.2.3}}{=} \sum_{\substack{s-r=c-a\\ s'-r'=c-b\\ t\in\mathbb{Z}_{\geq 0}\\ u\in\mathbb{Z}_{\geq 0}}} (-1)^{s+s'}\binom{a+s-s'}{r}\binom{r}{t}\binom{c-s'+t}{u}$$



$$= \sum_{\substack{s-r=c-a\\ s'-r'=c-b\\ t\in\mathbb{Z}_{\geq 0}\\ u\in\mathbb{Z}_{\geq 0}}} (-1)^{s+s'}\binom{a+s-s'}{r}\binom{r}{t}\binom{c-s'+t}{u}$$



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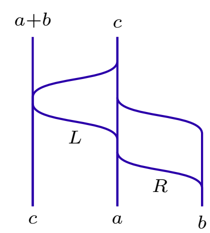
Now, fixing $L, R \in \mathbb{Z}_{\geq 0}$, we consider the coefficient $\kappa(L,R)$ of



in the sum above. Using the substitutions $A = L - R - a + c - s$, $B = c - s'$, we have

$$\begin{aligned}
\kappa(L,R) &= (-1)^{L+R+a}\sum_{B=0}^{b-R}\sum_{A=0}^{b-R}(-1)^{A+B}\binom{L-R-A+B}{B}\binom{L-R-A}{b-R-B}\binom{b-R}{A}\\
&= \sum_{B=0}^{b-R}\sum_{A=0}^{b-R}(-1)^{a+A+B}\binom{b-R}{B}\binom{L-R-A+B}{b-R}\binom{b-R}{A}\\
&= (-1)^{L+R+a}\sum_{B=0}^{b-R}(-1)^{B}\binom{b-R}{B}\sum_{A=0}^{b-R}(-1)^{A}\binom{L-R-A+B}{b-R}\binom{b-R}{A}.
\end{aligned}$$

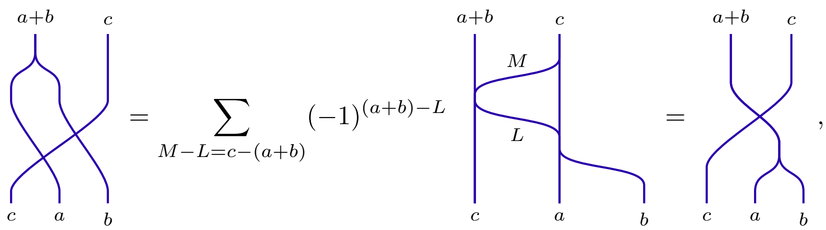
An application of Euler's finite difference theorem (cf. [4, (10.13)]), shows that

$$\sum_{A=0}^{b-R}(-1)^{A}\binom{L-R-A+B}{b-R}\binom{b-R}{A} = 1,$$

so

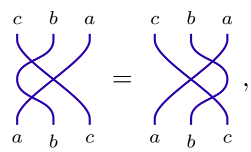
$$\kappa(L,R) = (-1)^{L+R+a}\sum_{B=0}^{b-R}(-1)^{B}\binom{b-R}{B} = (-1)^{L+a+b}\delta_{b,R},$$

by the binomial theorem. Therefore, we have



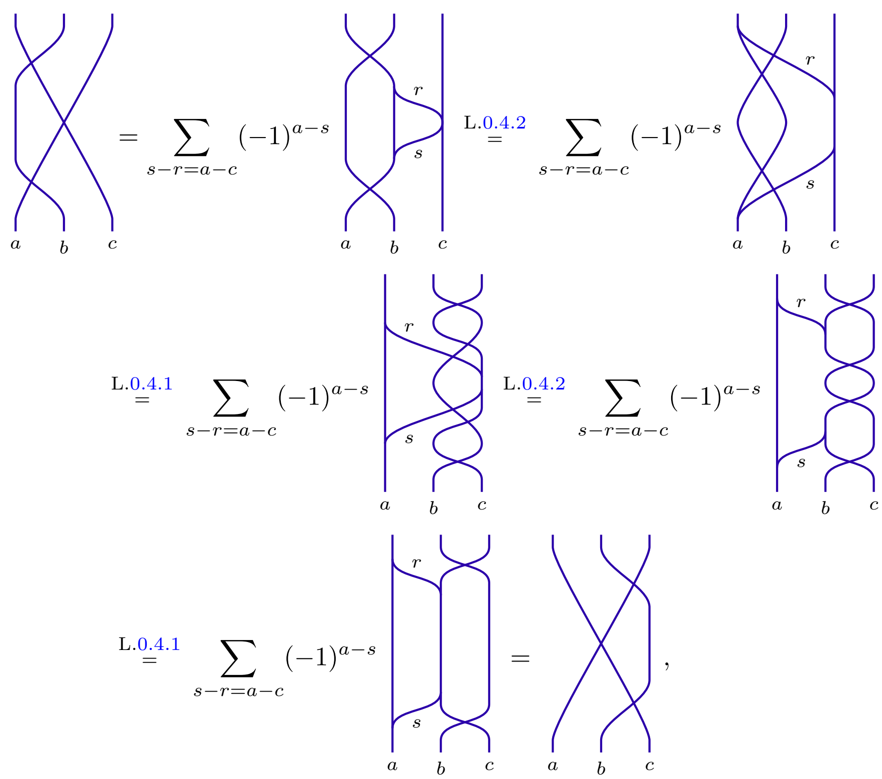
completing the proof. ☐

Lemma 0.4.3. *Crossings satisfy the braid relation:*



for all $a, b, c \in \mathbb{Z}_{\geq 0}$.

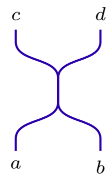
Proof. We have



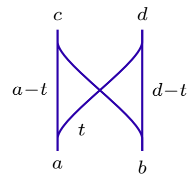
as desired. ☐

0.5. **A Useful Identity.** Using the crossing we record an identity in $\mathfrak{gl}$-**Web** which will be useful in later calculations.

Lemma 0.5.1. *For all $a, b, c, d \in \mathbb{Z}_{\geq 0}$ such that $a + b = c + d$, we have*

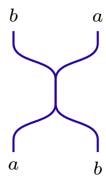


$$= \sum_{t \in \mathbb{Z}_{\geq 0}}$$

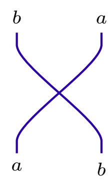


.

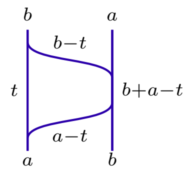
Proof. We go by induction on $n := a + b$. The base case $n = 0$ is trivial. Fix $n = a + b$ and assume the claim holds for all $a' + b' < n$. We first consider the case $b = c$. We have



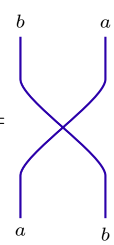
$$\overset{(0.4)}{=}$$



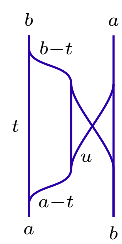
$$+ \sum_{t \in \mathbb{Z}_{>0}} (-1)^{t+1}$$



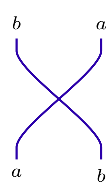
$$=$$



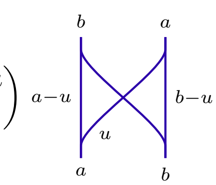
$$+ \sum_{\substack{t \in \mathbb{Z}_{>0} \\ u \in \mathbb{Z}_{\geq 0}}} (-1)^{t+1}$$



$$\overset{(0.1,0.2)}{=}$$



$$+ \sum_{\substack{t \in \mathbb{Z}_{>0} \\ u \in \mathbb{Z}_{\geq 0}}} (-1)^{t+1} \binom{a-u}{t}$$



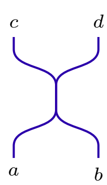
,

where we have applied the induction assumption in the second line. Then for fixed $u \leq a$, we have

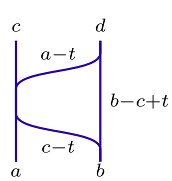
$$\sum_{t \in \mathbb{Z}_{>0}} (-1)^{t+1} \binom{a-u}{t} = \begin{cases} 0 & \text{if } u = a \\ 1 & \text{if } u < a, \end{cases}$$

giving the result.

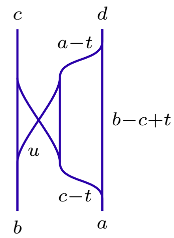
Now assume that $b > c$. Then we have



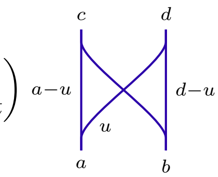
$$\overset{(0.2)}{=} \sum_{t \in \mathbb{Z}_{\geq 0}} \binom{c-b}{t}$$



$$= \sum_{t,u \in \mathbb{Z}_{\geq 0}} \binom{c-b}{t}$$



$$\overset{(0.1,0.2)}{=} \sum_{t,u \in \mathbb{Z}_{\geq 0}} \binom{c-b}{t}\binom{d-u}{b-c+t}$$



,

where we have used the induction assumption in the second line. Now, for a fixed $u \leq d$, we have

$$\sum_{t \in \mathbb{Z}_{\geq 0}} \binom{c-b}{t}\binom{d-u}{b-c+t} = 1,$$

by an application of the index shift formula (see [4, (6.69)]), which yields the result. □

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